USING THE SOLUTION TO THE REVERSE PROBLEM OF HEAT CONDUCTION IN THE CALCULATION OF THE HEAT TRANSFER COEFFICIENT FROM TEMPERATURE READINGS INSIDE THE BODY

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A method is proposed for determining the heat transfer coefficient from instantaneous temperature values at points inside a plate, a hollow cylinder, or a hollow sphere during heatup.

The coefficient of heat transfer at the surface of a hollow cylinder or a hollow sphere is defined as follows:

$$\mathrm{Bi}(\mathrm{Fo}) = \frac{1-k}{\theta - t(1, \mathrm{Fo})} \frac{\partial t}{\partial \rho}(1, \mathrm{Fo}). \tag{1}$$

This formula remains the same for a plate, with (1−k) replaced by 1 and variable ρ replaced by variable u. In order to determine the value of the Biot number Bi, therefore, it is necessary to know the temperature and the temperature gradient at the heated surface, i.e., the temperature distribution across the wall thickness. In some problems related to the determination of the heat transfer coefficient, a temperature measurement is technically very difficult at the heated surface but quite feasible at several internal points across the thickness. A relation between the temperatures at internal points and the temperature distribution across the wall (including the heated surface) can be established with the aid of the solution to the reverse heat conduction problem.

By solving the reverse heat conduction problem according to the method in [1, 2], a one-dimensional temperature field of bodies with a simple geometry can be expressed in terms of the temperatures at one or two points and of their time derivatives. Using this method, one arrives at the following expression for the temperature field of a plate, a hollow cylinder, or a hollow sphere heated at one surface (u = 1, ρ = 1) and ideally insulated at the other (u = 0, ρ = k):

$$t(\rho, \mathrm{Fo}) = \sum_{n=0}^{\infty} t^{(n)}(k, \mathrm{Fo}) P_n(\rho); \; t(u, \mathrm{Fo}) = \sum_{n=0}^{\infty} t^{(n)}(u, \mathrm{Fo}) P_n(u). \tag{2}$$

The radial polynomials $P_n(\rho)$ and $P_n(u)$ are determined with the aid of special relations which the author has derived by methods shown in [1] and put in a form more convenient for engineering calculations:

for a plate

$$P_n(u) = \frac{u^{2n}}{(2n)!}; \tag{3}$$

for a hollow sphere

$$P_n(\rho) = \frac{1}{(2n+1)!}\left(1 + \frac{2kn}{\rho}\right)\left(\frac{k-\rho}{k-1}\right)^{2n}; \tag{4}$$

for a hollow cylinder

$$P_0(\rho) = 1; \; P_1(\rho) = \frac{k^2}{4(k-1)^2}\left[\frac{\rho^2}{k^2} - 2\ln\frac{\rho}{k} - 1\right],$$

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$$P_2(\rho) = \frac{k^4}{64(k-1)^4}\left[\frac{\rho^4}{k^4} - 4\left(2\frac{\rho^2}{k^2} + 1\right)\ln\frac{\rho}{k} + 4\frac{\rho^2}{k^2} - 5\right]. \qquad (5)$$

Expressions for higher-order radial polynomials $P_n(\rho)$ for a hollow cylinder were given in [3].

When using expressions (2), one may consider that the values of $P_n(1)$ decrease fast with increasing n. Thus, in the case of a plate $P_1(1) = 1/2$, $P_2(1) = 1/24$, $P_3(1) = 1/720$. At an insulated surface of a solid body heated in any practical manner, on the other hand, the successively higher-order time derivatives of the temperature become also smaller in absolute value. All this makes it feasible, for approximate calculations of Bi(Fo), to break off the infinite sum in formula (2) after a finite number of terms $n = m$; in many cases $m = 2$ or 3 is sufficient.

The practical application of formula (2) is made more difficult by the necessity of calculating the time derivatives of the temperature at the surface $t^{(n)}(k, \mathrm{Fo})$. Indeed, if the temperature is measured within definite time intervals (as is usually done in practice), then determining its higher than first-order derivatives involves large errors, even when the most efficient methods of numerical differentiation (e.g., the method of least squares) are used.

Differentiations can be avoided if, besides the temperature of the insulated surface, one measures also the temperatures at internal points in the wall. It follows from expression (2) that the temperature at an arbitrary point ρ_i can be expressed in terms of the following approximate relation:

$$t(\rho_i, \mathrm{Fo}) = \sum_{n=0}^{m} t^{(n)}(k, \mathrm{Fo}) P_n(\rho_i). \qquad (6)$$

For a fixed number m, the unknown functions $t^n(k, \mathrm{Fo})$ with $n = 1, 2, \ldots, m$ can be determined from temperature readings at points inside the body, by solving the corresponding system of m algebraic equations (6). The uniqueness of the solution follows from the linear independence of functions $P_n(\rho)$.

If the number of temperature readings is less than m, then the missing equations can be made up for with the values of the first time derivatives of the temperatures which have been read $t'(\rho_i, \mathrm{Fo})$:

$$t'(\rho_i, \mathrm{Fo}) = \sum_{n=0}^{m-1} t^{(n+1)}(k, \mathrm{Fo}) P_n(\rho_i). \qquad (6a)$$

It is to be noted that a simultaneous use of expressions (6) and (6a) requires a certain amount of caution: it is necessary, for instance, to first examine the conditions under which the system of equations is solvable. An analysis has shown that in several cases (at definite ratios between coordinates ρ_i of the temperature test points) such a system may be either inconsistent or not fully determinate. Let us consider a plate whose temperature has been measured at two points with coordinates u_1 and u_2 respectively. The expression for the temperature distribution with $m = 2$ contains three unknowns, the determination of which requires that another equation be added to the two equations (6) set up for points u_1 and u_2. If relation (6a) for $t'(u_1)$ is used as that third equation, then $u_1^2 = u_2^2/5$ the system will be either inconsistent or not determinate enough (if condition $t'(u_1) = (t(u_2) - t(u_1))/2u_1^2$ is satisfied). If relation (6a) for $t'(u_2)$ is used as that third equation, however, then the system will have a unique solution for any ratio between u_1 and u_2 except the trivial $u_1 : u_2 = 1$. Thus, a preliminary solvability analysis allows us to choose from Eqs. (6a) those which together with Eqs. (6) will form a consistent and fully determinate system.

There is also another approach to solving the problem, which circumvents the need for a preliminary solvability analysis. From expressions (6) and (6a) one can obtain more equations than required for determining all unknowns. With all these equations, the system is overdeterminate and very often inconsistent. In the solution of engineering problems on the basis of test data, we are generally interested not in an exact answer but in the best answer possible under given conditions. In this case it is most convenient to solve the given system of equations by the method of least squares, which will clear any overdeterminacy or inconsistency. Proper suggestions can be found in [5], for example.

If the body wall is heated on both sides, then the minimum number of temperature readings is 2 and, instead of expression (2), we have the following relation which defines the temperature field in terms of measured temperatures $t_1(\mathrm{Fo})$ and $t_2(\mathrm{Fo})$ with their derivatives [2]:

$$t(\rho, \mathrm{Fo}) = \sum_{n=0}^{\infty} t_1^{(n)}(\mathrm{Fo}) P_{1n}(\rho) + \sum_{n=0}^{\infty} t_2^{(n)}(\mathrm{Fo}) P_{2n}(\rho). \qquad (7)$$

The expressions for the radial polynomials $P_{1n}(\rho)$ and $P_{2n}(\rho)$ for this case were given in [2].

TABLE 1. Calculation of the Heat Transfer Coefficients

Fo	$Bi/1-k=1{,}0$	$Bi/1-k=5{,}0$
0,0313	1,049	4,119
0,0782	1,219	4,947*
0,1565	1,113	5,413
0,313	1,019*	4,559
0,782	1,029	4,814
1,565	0,973*	4,926*

Note. The Bi *values were calculated from the temperature at points ρ = 0.9, 0.6, 0.2, the other values were calculated from the temperatures at points ρ = 0.9, 0.8, 0.2.

An analogous procedure can be used for expressing the derivatives $t^n(Fo)$ in formula (7) in terms of temperature readings at internal points ρ_i. These points ρ_i must lie between the hot surface and the nearest point at which temperature t_1 or t_2 is measured. When two surfaces of a body are heated, then, of course, the number of internal temperature readings necessary for calculating $t_1^n(Fo)$ and $t_2^n(Fo)$ is double the number needed in the previous case of one heated surface.

System (7), just as system (6), has a simple solution: this follows from the linear independence of all functions $P_{1n}(\rho)$ and $P_{2n}(\rho)$.

In order to reduce the number of measurements at internal points, one may use, in addition to temperatures, also their first derivatives at given points. In that case the system of equations must be first tested for consistency or it must be solved by the method of least squares.

Thus, we can find relations which will yield the temperature at any point across the wall of a body – including the heated surface. Expressions $\partial t/\partial \rho$ are analogous to (2) or (7), with $P_n(\rho)$ replaced by $(dP_n/d\rho)(\rho)$.

The Biot number was calculated for a hollow cylinder with the inside surface insulated, as shown in Table 1, first from the temperature readings at two internal points and at the insulated surface ($k = 0.2$), then according to the exact solution [4] with $\theta = 1$ and a zero initial temperature field. The accuracy of the results is adequate for use in engineering designs.

NOTATION

θ	is the ambient temperature;
x, r	are the space coordinates;
L	is the thickness of a plate;
R	is the radius of a surface;
$\rho = r/R_h$;	
$k = R_i/R_h$;	
$u = x/L$;	
a	is the thermal diffusivity;
λ	is the thermal conductivity;
α	is the heat transfer coefficient;
τ	is the time;
$Fo = a\tau/L^2$	is the Fourier number;
$Fo = a\tau/(R_{out}-R_{in})^2$	is the Fourier number;
$Bi = \alpha L/\lambda$	is the Biot number;
$Bi = \alpha(R_{out}-R_{in})/\lambda$	is the Biot number;
$t^n(Fo)$	is the n-th order derivative of temperature with respect to Fo.

Subscripts

out	denotes the outside surface;
in	denotes the inside surface;
i	denotes the insulated surface;
h	denotes the heated surface.

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